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THE MORSE INDEX THEOREM WHERE THE ENDS ARE SUBMANIFOLDS

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ABSTRACT. In this paper the Morse Index Theorem is proven in the case where submanifolds P and Q are at the endpoints of a geodesic, γ. At γ, the index of the Hessian of the energy function defined on paths joining P and Q is computed using P-focal points, and a calculation at the endpoint of γ, involving the second fundamental form of Q.

1. Introduction. Let M be a complete Riemannian manifold with submanifolds P and Q. The energy function E is defined on the space $\Omega(M;P,Q)$ of piecewise C^∞ paths joining P and Q. A path $\gamma \in \Omega(M;P,Q)$ is a critical point for E when γ is a geodesic intersecting P and Q orthogonally. The tangent space, $T\Omega_\gamma$, consists of piecewise C^∞ vector fields along γ with initial and final vectors tangential to P and Q, respectively. A symmetric bilinear map, I, is defined on $T\Omega_\gamma \times T\Omega_\gamma$ to R and is called the Morse index form.

When Q is a point, the Morse Index Theorem yields the index of I as the sum of the P-focal points along γ counted with multiplicities. Both Ambrose [**1**] and Bolton [**2**] have proven index theorems in the general case, where Q is a submanifold. Ambrose defines a "(P,Q) conjugate point," while Bolton uses the notion of a signed (P,Q) focal point which is employed in the calculations of the index of I.

In this paper the index of a critical point γ is found using P-focal points and a computation at the endpoint of γ contained in Q, involving the second fundamental form of Q with respect to $\dot\gamma$. This method allows for a simpler proof of the theorem as well as an easy computation of the index of geodesics in many spaces. In a paper to follow this one, the homotopy type of some path spaces joining submanifolds on a Riemannian manifold have been computed. This result is obtained, with some minor modifications, by following the proof in Milnor [**4**, pp. 88–95].

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2. Definitions. M is a complete Riemannian manifold of dimension d with the Levi-Civita connection.

$\gamma(t)$, $t \in [0,T]$, is a geodesic in M.

P and Q are submanifolds of M with $\gamma(0) \in P$; $\gamma'(0) \perp P_{\gamma(0)}$; $\gamma(T) \in Q$; $\gamma'(T) \perp Q_{\gamma(T)}$.

r is the dimension of $Q_{\gamma(T)}$, $0 \leq r < d$.

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H is the linear space of continuous piecewise C^∞ vector fields along γ which are orthogonal to γ and whose initial and final vectors are in $P_{\gamma(0)}$ and $Q_{\gamma(T)}$, respectively. Then,

$$H = \left\{ V(t) = \sum_{i=1}^{d-1} h_i(t)E_i(t) \text{ with } V_{(0)} \in P_{\gamma(0)}; V(T) \in Q_{\gamma(T)} \right\},$$

where $E_1, \ldots, E_{d-1}$ are orthonormal parallel vector fields along γ and orthogonal to γ, and $h_1, \ldots, h_{d-1}$ are real valued continuous piecewise C^∞ functions defined on $[0, T]$.

A *Jacobi field* X is a vector field along γ which satisfies the differential equation $X'' - RX = 0$, where $RX = R(\gamma', X)\gamma'$ is the curvature tensor of the Levi-Civita connection.

A *P-Jacobi field* is a Jacobi field which is orthogonal to γ with $J(0) \in P_{\gamma(0)}$ and $J'(0) - S_0 J(0) \perp P_{\gamma(0)}$, where S_0 is the second fundamental form of P at $\gamma(0)$ with respect to $\gamma'(0)$.

$J_1, \ldots, J_{d-1}$ are $d-1$ linearly independent P-Jacobi fields which span the space of P-Jacobi fields.

B is the set of all $X \in H$ such that

$$X(t) = \sum_{i=1}^{d-1} f_i(t)J_i(t) \quad \text{with } X(T) = 0,$$

and $f_1, \ldots, f_{d-1}$ are real valued continuous piecewise C^∞ functions defined on $[0, T]$.

I is a symmetric bilinear map from $H \times H$ to R defined as follows.

$$I(X,Y) = \int_0^T \langle RX(t) - X''(t), Y(t)\rangle\, dt + \sum_i \langle X'(p_i^-) - X'(p_i^+), Y(p_i)\rangle + \langle X'(t) - S_t X(t), Y(t)\rangle|_0^T$$

where p_i is a point of discontinuity of X' in $(0,T)$; S_0 is the second fundamental form of P at $\gamma(0)$ with respect to $\gamma'(0)$; and S_T is the second fundamental form of Q at $\gamma(T)$ with respect to $\gamma'(T)$.

A *P-focal point* is a point $\gamma(t)$, $t \in (0, T]$, for which there exists a nonzero P-Jacobi field which vanishes at t.

The *multiplicity*, m, of the P-focal point $\gamma(t)$ is the dimension of the space of P-Jacobi fields which vanish at t.

A is a symmetric bilinear map defined on the space spanned by $J_1, \ldots, J_{d-1}$ whose value at T is contained in $Q_{\gamma(T)}$ and is defined as follows:

$$A(V, W) = \langle V'(T) - S_T V(T), W(T)\rangle.$$

3. The Index Theorem.

THEOREM (MORSE INDEX THEOREM WITH VARIABLE ENDPOINTS). *The index of I is equal to the number of points $\gamma(t)$, with $0 < t < T$, such that $\gamma(t)$ is a P-focal point; each such P-focal point counted with its multiplicity, plus the index of A. (Assume $\gamma(T)$ is not a P-focal point.)*

The above theorem states

$$i(I) = \sum_{i=1}^{k} m_i + i(A),$$

when $\gamma(T)$ is not a P-focal point, where $i(I)$ = index of I; $i(A)$ = index of A; m_i is the multiplicity of $\gamma(t_i)$; and $\gamma(t_1), \ldots, \gamma(t_k)$ are the set of P-focal points along γ, $(0 < t_1 < \cdots < t_k < T)$.

Our aim will be to write $H = B \oplus B^C$, where I is positive on B. (For definition of B see §2.) We will show that B^C is a finite dimensional space and construct a subspace of B^C on which I is negative definite and whose dimension is equal to or greater than any other subspace of H on which I is negative definite. This will yield $i(I)$.

REMARK. The subspace B can be characterized as the set of all vector fields in H whose values at the P-focal points are in the span of $J_1, \ldots, J_{d-1}$ and whose value at T is zero.

This follows from the fact that any broken C^∞ vector field which can be expressed as $\sum_{i=1}^{d-1} f_i(t)E_i(t)$ and is in the span of $J_1(t_i), \ldots, J_{d-1}(t_i)$ at all P-focal points t_i can also be written as $\sum_{i=1}^{d-1} g_i(t)J_i(t)$ for broken C^∞ functions g_i [**3**, p. 231].

The next two definitions will yield $(\sum_{i=1}^{k} m_i + r)$ linearly independent elements of H whose span will be denoted by B^C.

DEFINITION OF K_2. Since T is not a P-focal point, we can choose r linearly independent P-Jacobi fields $K_1, \ldots, K_r$ with the following properties: (1) $K_1(T), \ldots, K_r(T)$ span $Q_{\gamma(T)}$; (2) A is negative definite on the span of $K_1, \ldots, K_N$, where N = index A and $N \leq r$; and (3) A is positive (≥ 0) on the span of $K_{n+1}, \ldots, K_r$. (Recall that A was defined as a symmetric bilinear map on those P-Jacobi fields which, when evaluated at T, lie in $Q_{\gamma(T)}$. The dimension of this space is r.)

DEFINITION OF $V_i^{j_i}$. Consider the P-focal point $\gamma(t_i)$ with multiplicity m_i. Let $Y_i^1, \ldots, Y_i^{m_i}$ be m_i linearly independent P-Jacobi fields such that

(1) $Y_i^{j_i}(t_i) = 0$ for $j_i = 1, \ldots, m_i$,

(2) $\{Y_i'^{j_i}(t_i)\}$ for $j_i = 1, \ldots, m_i$, form an orthonormal set.

Let $\tilde{Z}_i^{j_i}$ be parallel vector fields along γ such that

$$\tilde{Z}_i^{j_i}(t_i) = -Y_i'^{j_i}(t_i) \quad \text{for } j_i = 1, \ldots, m_i.$$

Let $\phi_i\colon [0,T] \to R$ be a C^∞ function such that (1) $\phi_i(t_i) = 1$; (2) $\phi_i(t_i)$ has small support about t_i; and (3) $0 \leq \phi_i(t) \leq 1$. Let

$$Z_i^{j_i}(t) = \phi_i(t)\tilde{Z}_i^{j_i}(t); \qquad i = 1, \ldots, m_i.$$

Let

$$V_i^{j_i}(t) = \begin{cases} Y_i^{j_i}(t) + \lambda Z_i^{j_i}(t) & \text{for } 0 \leq t \leq t_i, \\ \lambda Z_i^{j_i}(t) & \text{for } t_i \leq t \leq T, \end{cases}$$

where $\lambda > 0$.

(*Note*: It follows from (1) and (2) in the definition of $V_i^{j_i}$ that

$$(\text{span}[Y_i'^1(t_i), \ldots, Y_i'^{m_i}(t_i)])^\perp = \text{span}(J_1(t_i), J_2(t_i), \ldots, J_{d-1}(t_i), \gamma'(t_i)).)$$

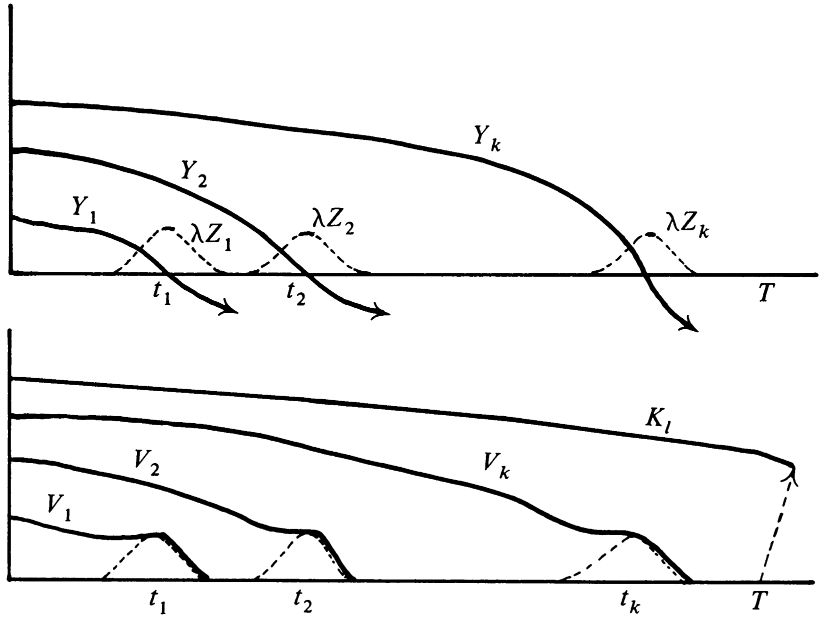

FIGURE 1

DEFINITION OF B^C. Let B^C denote the span of the vectors $V_i^{j_i}$ ($i = 1, \ldots, k$; $j_i = 1, \ldots, m_i$) and K_l ($l = 1, \ldots, r$).

CLAIM. (1) The dimension of B^C is $\sum_{i=1}^k m_i + r$,

(2) $B^C \cap B = 0$.

The claim follows from the definitions of B and B^c, by making the support of each ϕ_i small enough, and by looking at Figure 1.

LEMMA 1. $H = B \oplus B^c$.

PROOF. Let $x \in H$. There is a $c \in B^c$ such that $x - c \in B$. This can be accomplished by choosing c equal to a linear combination of elements in B^c, so that at P-focal point the value of $x - c$ lies in the span of the P-Jacobi fields and at T is equal to zero. Thus $x - c \in B$. Since $B \cap B^c = 0$, we have Lemma 1.

Lemma 2 will show I is positive on B and Lemma 3 will exhibit a subspace of B^C on which I is negative definite.

LEMMA 2. *$I(V, V) \geq 0$ for $V \in B$.*

PROOF. Let $V \in B$. Then

$$V = \sum_{i=1}^{d-1} f_i J_i$$

and

$$\begin{aligned} I(V,V) &= \left\langle \sum_{i=1}^{d-1} f_i(T)J_i'(T) - S_T[V(T)], V(T) \right\rangle \\ &\quad + \int_0^T \left\langle \sum f_i'(t)J_i(t), \sum f_i'(t)J_i)t) \right\rangle dt \\ &= \int_0^T \left\langle \sum f_i'(t)J_i(t), \sum f_i'(t)J_i(t) \right\rangle dt \quad (\text{since } V(T)=0) \\ &\geq 0. \end{aligned}$$

For the first equality, see [**3**, p. 229]. This proves Lemma 2.

LEMMA 3.

$$\operatorname{index}(I_{|B^C}) = \sum_{i=1}^{k} m_i + \operatorname{index}(A).$$

PROOF. $K_1, \ldots, K_r$ were chosen to be r linearly independent P-Jacobi fields such that A is negative definite (< 0) on the span of $K_1, \ldots, K_N$ and positive (≥ 0) on $K_{N+1}, \ldots, K_r$, $N = \operatorname{index} A$.

We wish to show that I is negative definite on the span of $\{V_i^{j_i}\}_{i=1,\ldots,k;\ j_i=1,\ldots,m_k}$ and $K_1, \ldots, K_N$, and that I is positive on the span of $K_{N+1}, \ldots, K_r$.

$$\begin{aligned} I&\left(\sum_{i,j_i} \alpha_i^{j_i} V_i^{j_i} + \sum_{l=1}^{N} \beta_l K_l\right) \\ &= I\left(\sum \alpha_i^{j_i} V_i^{j_i}\right) && (1) \\ &\quad + 2I\left(\sum_{i,j_i} \alpha_i^{j_i} V_i^{j_i}, \sum_{l=1}^{N} \beta_l K_l\right) && (2) \\ &\quad + I\left(\sum_{l=1}^{N} \beta_l K_l\right). && (3) \end{aligned}$$

The computation of (3) yields $I(\sum \beta_l K_l) < 0$.

PROOF. Let

$$K = \sum_{l=1}^{N} \beta_l K_l.$$

Then

$$\begin{aligned} I(K) = \int_0^T \langle RK - K'', K \rangle\, dt &+ \sum_{\substack{\text{jumps} \\ \text{of } K'}} \langle K'(p_i^-) - K'(p_i^+), V_i^{j_i}(p_i) \rangle \\ &+ \langle K'(t) - S_t K(t), K(t) \rangle |_0^T = \langle K'(T) - S_T K(T), K(T) \rangle. \end{aligned}$$

This follows from the fact that K is a P-Jacobi field which is smooth and satisfies $RK - K'' = 0$ and $K'(0) - S_0 K(0) \perp P_{\gamma(0)}$.

So $I(K) = \langle K'(T) - S_T K(T), K(T) \rangle = A(K) < 0$, since A is negative definite on the span of $K_1, \ldots, K_N$.

The computation of (2) yields

$$I\left(\sum \alpha_i^{j_i} V_i^{j_i}, \sum \beta_l K_l\right) = 0.$$

PROOF. When (2) is expanded we get linear combinations of terms of the form $I(V_i^{j_i}, K_l)$.

$$I(V_i^{j_i}, K_l) = \int_0^T \langle RK_l - K_l'', V_i^{j_i}\rangle\, dt + \sum_{\substack{\text{jumps}\\ \text{of } K'}} \langle K'(p_i^-) - K'(p_i^+), K(p_i)\rangle$$
$$+ \langle K_l'(t) - S_t K_l(t), V_i^{j_i}(t)\rangle|_0^T = 0,$$

since K_l is a P-Jacobi field, $K_l'(0) - S_0 K(0) \perp P_{\gamma(0)}$ $(V_i^{j_i}(0) \in P_{\gamma(0)})$, and $V_i^{j_i}(T) = 0$.

The computation of (1) yields $I(\sum \alpha_i^{j_i} V_i^{j_i}) < 0$.

PROOF. Let

$$\hat{Y}_i^j = \begin{cases} Y_i^j(t), & 0 \le t \le t_i, \\ 0, & t_i \le t \le T. \end{cases}$$

Then $V_i^j(t) = \hat{Y}_i^j(t) + \lambda Z_i^j(t); t \in [0, T]$.

$$I(V_i^j, V_h^l) = I(\hat{Y}_i^j + \lambda Z_i^j, \hat{Y}_h^l + \lambda Z_h^l)$$
$$= \begin{cases} I(\hat{Y}_i^j, \hat{Y}_h^l) & \text{(a)} \\ + \lambda^2 I(Z_i^j, Z_h^l) & \text{(b)} \\ + \lambda I(Z_i^j, \hat{Y}_h^l) + \lambda I(\hat{Y}_i^j, Z_h^l) & \text{(c)}. \end{cases}$$

$$\text{(a)} = I(\hat{Y}_i^j, \hat{Y}_h^l) = \int_0^T \langle R\hat{Y}_i^j - \hat{Y}_i''^j, \hat{Y}_h^l\rangle\, dt + \langle \hat{Y}_i'^j(t_i^-) - \hat{Y}_i'^j(t_i^+), \hat{Y}_h^l(t_i)\rangle$$
$$+ \langle \hat{Y}_i'^j - S_t \hat{Y}_i^j, \hat{Y}_h^l\rangle|_0^T = 0, \quad \text{when } h \le i.$$

This follows from $\hat{Y}_i''^j - R\hat{Y}_i^j = 0$, $\hat{Y}_h^l(t_i) = 0$, $\hat{Y}_h^l(T) = 0$, and $\hat{Y}_i'^j(0) - S_0 \hat{Y}_i^j(0) \perp P_{\gamma(0)}$. Since $I(\hat{Y}_i^j, \hat{Y}_h^l) = I(\hat{Y}_h^l, \hat{Y}_i^j)$, the same argument works for $i \le h$.

(c) $= -2\lambda \delta_{ih}\delta_{jl}$, which is shown to be true as follows. For $i \ne h$, we get

$$I(Z_i^j, \hat{Y}_h^l) = \int_0^T \langle R\hat{Y}_h^l - \hat{Y}_h''^l, Z_i^j\rangle\, dt + \langle \hat{Y}_h'^l(t_h^-) - \hat{Y}_h'^l(t_h^+), Z_i^j(t_h)\rangle$$
$$+ \langle \hat{Y}_h'^l(t) - S_t \hat{Y}_h^l(t), Z_i^j(t)\rangle|_0^T = 0,$$

since $\hat{Y}_h^l$ is a P-Jacobi field on $[0, t_h]$,

$$Z_i^j(0) = Z_i^j(T) = 0 \quad \text{and} \quad Z_i^j(t_h) = 0$$

when the support of ϕ_i is small enough.

For $i = h$, we get the same as above except at t_i,

$$\begin{aligned} I(Z_i^j, \hat{Y}_i^l) &= \langle \hat{Y}_i'^l(t_i^-) - \hat{Y}_i'^l(t_i^+), Z_i^j(t_i)\rangle \\ &= \langle \hat{Y}_i'^l(t_i^-), Z_i^j(t_i)\rangle \quad (\text{since } \hat{Y}_i^l(t) = 0 \text{ for } t \in [t_i, T]) \\ &= \langle Y_i'^l(t_i), -Y_i'^j(t_i)\rangle \\ &= -\delta_{jl}\|Y_i'^j(t_i)\|^2 \\ &\qquad (\text{since } \{Y_i'^1, Y_i'^2, \dots, Y_i'^{m_i}\} \text{ is an orthonormal set evaluated at } t_i) \\ &= -\delta_{jl}. \end{aligned}$$

Therefore, $\lambda I(Z_i^j, \hat{Y}_h^l) + \lambda I(\hat{Y}_i^j, Z_h^l) = -2\lambda\delta_{ih}\delta_{jl}$.

Putting together the results from (a), (b), and (c), we have

$$I(V_i^j, V_h^l) = \lambda^2 I(Z_i^j, Z_h^l) - 2\lambda\delta_{ih}\delta_{jl}.$$

Notation. Let A be the $(\sum m_i) \times (\sum m_i)$ symmetric matrix $(I(Z_i^{j_i}, Z_k^{j_k}))$. If $X = \sum a_i^{j_i} V_i^{j_i}$, $Y = \sum b_i^{j_i} V_i^{j_i}$ let

$$\langle X, Y\rangle = \sum a_i^{j_i} b_i^{j_i} \quad \text{and} \quad \|X\|^2 = \langle X, X\rangle.$$

Then for $X = \sum_{i,j_i} a_i^{j_i} V_i^{j_i}$ we have

$$I(X,X) = \lambda^2\langle AX, X\rangle - 2\lambda\langle X, X\rangle.$$

If $A = 0$, $I(X,X) < 0$ for $\lambda > 0$, $X \neq 0$. If $A \neq 0$, $\|A\| \neq 0$, let $0 < \lambda < 2/\|A\|$. Then

$$\begin{aligned} I(X,X) &= \lambda^2\langle AX, X\rangle - 2\lambda\langle X, X\rangle \leq \lambda^2\|AX\|\,\|X\| - 2\lambda\|X\|^2 \\ &\leq \lambda^2\|A\|\,\|X\|^2 - 2\lambda\|X\|^2 \\ &< \lambda\frac{2}{\|A\|}\|A\|\,\|X\|^2 - 2\lambda\|X\|^2 = 0. \end{aligned}$$

This gives

$$I\left(\sum \alpha_i^{j_i} V_i^{j_i}, \sum \alpha_i^{j_i} V_i^{j_i}\right) < 0 \quad \text{for } \sum \alpha_i^{j_i} V_i^{j_i} \neq 0.$$

Thus we have the computation of (1).

The results from (1), (2), and (3) show that I is negative definite on the span of $\{V_i^{j_i}, K_1, \dots, K_N\}_{i=1,\dots,k;\ j_i=1,\dots,m_i}$.

In order to finish proving Lemma 3 we need to show I is positive (≥ 0) on the span of $K_{N+1}, \dots, K_r$.

$$I\left(\sum_{l=N+1}^{r} b_l K_l\right) = A\left(\sum_{l=N+1}^{r} b_l K_l\right) \geq 0,$$

since A is positive on the span of $K_{N+1}, \dots, K_r$ and $\sum b_l K_l$ is a P-Jacobi field. This proves Lemma 3.

DEFINITION OF B_-^c, B_+^c. Let

$$B_-^c = \mathrm{Span}(K_1, \dots, K_N) \qquad \text{and} \qquad B_+^c = \mathrm{Span}(K_{N+1}, \dots, K_r).$$

From Lemma 3 we have that $I|_{B_-^c} < 0$ and $I|_{B_+^c} \geq 0$, while from Lemma 2 we have that $I|_B \geq 0$.

Write $H = B \oplus B_+^c \oplus B_-^c$. In order to complete the Index Theorem we need to show that $I(B, B_+^c) = 0$, which will follow if

$$I\left(K_j, \sum f_i J_i\right) = 0 \quad \text{for } j = N+1, \dots, r,\ i = 1, \dots, d-1.$$

This is true since K_j is a P-Jacobi field, $S_0 K_j(0) - K_j'(0) \perp (\sum f_i J_i)(0)$ and $(\sum f_i J_i)(T) = 0$.

We therefore have a subspace B_-^c of H on which I is negative definite and whose dimension $\sum_{i=1}^k m_i + i(A)$ is the maximum value I can attain on any subspace of H. Thus the Index Theorem is proven.

4. Remarks. 1. The proof of the Index Theorem is symmetric with respect to the P and Q submanifolds at the ends of the geodesic. That is, we can use Q-Jacobi fields and Q-focal points to prove the theorem.

2. If $\gamma(T)$ is a P-focal point, then the proof of the Index Theorem is still valid when

(a) $Q_{\gamma(T)}$ is contained in the span of the P-Jacobi fields or when

(b) $\gamma(0)$ is not a Q-focal point, or if $P_{\gamma(0)}$ is contained in the span of the Q-Jacobi fields.

REFERENCES

1. W. Ambrose, *The Index Theorem in Riemannian geometry*, Ann. of Math. (2) **73** (1961), 49–86.
2. John Bolton, *The Morse Index Theorem in the case of two variable end-points*, J. Differential Geom. **12** (1977), 567–581.
3. Richard L. Bishop and Richard L. Crittenden, *Geometry of manifolds*, Academic Press, New York, 1964.
4. J. Milnor, *Morse theory*, Ann. of Math. Studies, no. 51, Princeton Univ. Press, Princeton, N.J., 1973.

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